Lecture 9 (Counting Techniques)

How do you count the number of people in a crowded room? You could count heads, since for each person there is exactly one head. Alternatively, you could count ears and divide by two. Of course, you might have to adjust the calculation if someone lost an ear in a pirate raid or someone was born with three ears. The point here is that you can often count one thing by counting another, though some fudge factors may be required. This is a central theme of counting, from the easiest problems to the hardest.
Let us note that **every counting problem comes down to determining the size of some set.**

We first present basic counting rules. Then we will show how they can be used to solve many different counting problems.

The product rule: Suppose a task has $n \in \mathbb{N}$ **compulsory** parts and the i-th part can be completed in $m_i \in \mathbb{N}$ ways for $i = 1, 2, \ldots, n$. Then the task can be completed in $m_1 m_2 \ldots m_n$ ways.
In terms of sets, if $A_1, A_2, \ldots, A_n$ are sets, then

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1|.|A_2|.\ldots.|A_n|.$$

1. How many three digit natural numbers can be formed using digits $0, 1, \ldots, 9$?

   Solution: $9 \times 10 \times 10$ ways.

2. The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

   Solution: 100×26 ways.

3. How many functions are there from a set with m elements to a set with n elements?

   Solution: $n \times n \times \ldots \times n (m \text{ times})$, that is, n^m.

4. How many one-to-one functions are there from a set with m elements to one with n elements?

   Solution: $n \times (n-1) \times (n-2) \times \ldots \times (n-m+1)$.

5. Let $|S| = n$. $|P(S)| = 2^n$.

   Solution: consider the one-to-one correspondence between subsets of S and bit strings (each element takes on a value of 0 or 1) of length $|S|$. A susbset of S is associated with the bit string with a 1 in the ith position if the ith element in the list is in the subset. By the multiplication rule, there are $2^{|S|}$ bit strings of length $|S|$. Therefore, $|P(S)| = 2^{|S|}$.

The sum rule: Suppose a task consists of n **alternative** parts (either parts), and the i-th part can be completed in m_i ways, $i = 1, \ldots, n$. Then the task can be completed in $m_1 + m_2 + \ldots + m_n$

ways. Following examples illustrate the rule.
In terms of sets, if $A_1, A_2, \ldots, A_n$ are disjoint sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

1. Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

   Solution: 37+83= 110.

2. How many three digit natural numbers with distinct digits can be formed using digits $1, \ldots, 9$ such that each digit is odd or each digit is even?

   Solution: The task has two alternative parts. Part 1: form a three digit number with distinct digits using digits from $\{1, 3, 5, 7, 9\}$. Part 2: form a three digit number with distinct digits using digits from $\{2, 4, 6, 8\}$. Observe that Part 1 is a task having three compulsory subparts. Using multiplication rule, we see that Part 1 can be done in $5 \times 4 \times 3$ ways. Part 2 is a task having three compulsory subparts. So, it can be done in $4 \times 3 \times 2$ ways. Since our task has alternative parts, addition rule implies $60 + 24 = 84$.

The subtraction rule (The inclusion-exclusion principle): If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
In terms of sets, if A and B are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

The subtraction rule is also known as the principle of inclusion-exclusion, especially when it is used to count the number of elements in the union of two sets.

1. How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

   Solution: $2^7 + 2^6 - 2^5$.

2. A computer company receives 350 applications from computer graduates for a job. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

   Solution: Let A_1 be the set of students who majored in computer science and A_2 the set of students who majored in business. By the subtraction rule, the number of students who majored either in computer science or in business equals $A \cup B$

   $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316$. Thus, $350 - 316 = 34$ of the applicants majored neither in computer science nor in business.

Pigeonhole Principle: If $n+1$ pigeons (resp. objects) are distributed into n holes (resp. boxes), then some hole (box) must contain at least 2 of the pigeons (objects).

Proof: Assume $n+1$ pigeons are distributed into n boxes. Suppose the boxes are labeled $B_1, B_2, \ldots, B_n$, and assume that no box contains more than 1 object. Let k_i denote the number of objects placed in B_i. Then $k_i \leq 1$ for $i = 1, \ldots, n$, and so $k_1+k_2+\ldots+k_n \leq 1+1+\ldots+1 \leq n$. But this contradicts the fact that $k_1 + k_2 + \ldots + k_n = n+1$, the total number of objects we started with.

1. Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
2. In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.
3. How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

   Solution: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects, where $\lceil . \rceil$ denotes the ceiling function. In terms of functions, If $|X| > k|Y|$, then every function $f : X \to Y$ maps at least $k+1$ different elements of X to the same element of Y.

1. Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.
2. Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

   Solution: Let us write each of the $n+1$ integers $a_1, a_2, \ldots, a_{n+1}$ as a power of 2 times an odd integer. In other words, let $a_j = 2^{k_j} q_j$ for $j = 1, 2, \ldots, n+1$, where k_j is a non-negative integer and q_j is odd. The integers $q_1, q_2, \ldots, q_{n+1}$ are all odd positive integers less than $2n$. Because there are only n odd positive integers less than $2n$, it follows from the pigeonhole principle that two of the integers $q_1, q_2, \ldots, q_{n+1}$ must be equal. Therefore, there are distinct integers i and j such that $q_i = q_j$. Let q be the common value of q_i and q_j. Then, $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$. It follows that if $k_i < k_j$, then a_i divides a_j, while if $k_i > k_j$, then a_j divides a_i.

Permutations A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an r-permutation.

Example: Let $S = \{1, 2, 3\}$. Then the arrangement $3, 1, 2$ is a 3-permutation and the arrangement $3, 2$ is a 2-permutation of S.

Theorem: The number of r-permutations of a set with n distinct elements is

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1).$$

Combination An r-combination of elements of a set is an unordered selection from the set. Thus an r-combination is simply a subset of the set with r elements.

Theorem: The number of r-permutations of a set with n elements is

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

Permutation with repetition Consider the following counting problem when repetition is allowed.

Example: How many strings of length n can be formed using English alphabets?
Solution: By the product rule: 26^n.

Theorem: The number of r-permutations of a set with n elements with repetition allowed is n^r.

Theorem: There are $C(n+r-1, r)$ r-combinations from a set with n elements when repetition of elements is allowed.